Phase space behavior of Laplace Eigenfunctions

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Overview of of the talk

- What are eigenfunctions and how do they help model physical processes on surfaces?
- What did I contribute to our understanding of eigenfunctions?

The second derivative operator

Second derivative operator

Operator $\frac{d^2}{dx^2}$ takes function f to second derivative

$$\left(\frac{d^2}{dx^2} f\right)(x) = f''(x)$$

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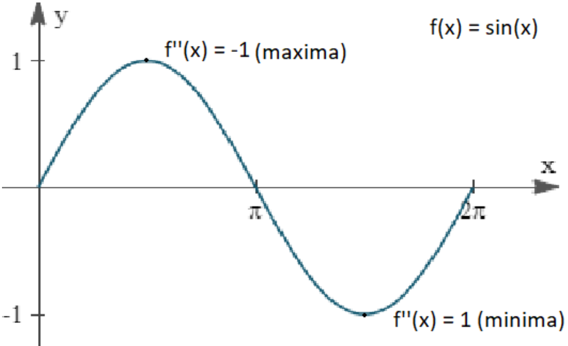
$$\left(\frac{d^2}{dx^2}f\right)(x) = f''(x)$$

$f''(x)$ says to what "degree" x is a minima



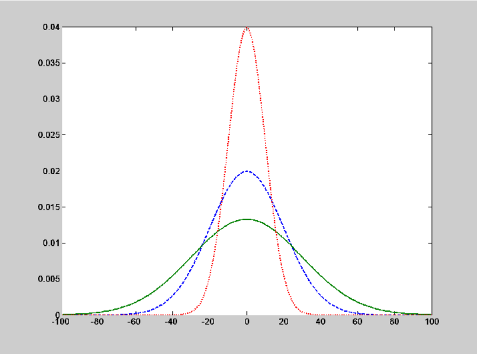
The second derivative operator

$\frac{d^2}{dx^2}$ is ubiquitous in physical modeling

Heat Diffusion

$u(x,t)$ is the temperature at x at time t and evolves according to

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$



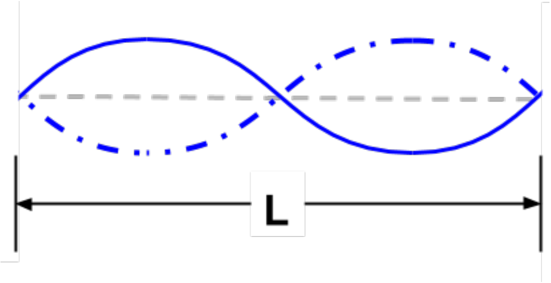
The second derivative operator

$\frac{d^2}{dx^2}$ is ubiquitous in physical modeling

Wave Propagation

$u(x,t)$ is the height of a wave at x at time t and evolves according to

$$\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$



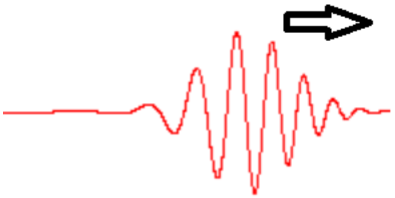
The second derivative operator

Focus on probability distribution of location of quantum particle

Quantum Particle

$|u(x,t)|^2$ is the probability distribution of locating a particle at position x and time t which evolves according to

$$i\frac{\partial}{\partial t}u(x,t) = -\frac{\partial^2}{\partial x^2}u(x,t)$$



Eigenvalues, eigenvectors, and eigenfunctions

- *Eigenfunctions* generalize eigenvectors. Help solve differential equations.

Eigenvalues, eigenvectors, and eigenfunctions

Eigenvectors of matrices

M is an $n \times n$ matrix, v is an $n \times 1$ vector and λ is a real number

$$Mv = \lambda v$$

- v is called a *eigenvector* with *eigenvalue* λ

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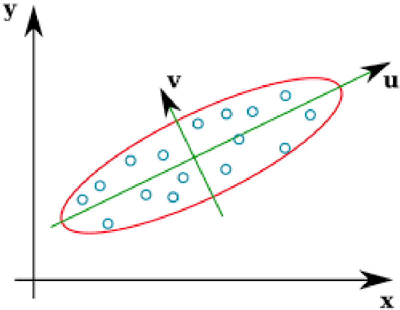
Eigenvectors identify "principal directions"



Eigenvalues, eigenvectors, and eigenfunctions

Eigenfunctions of $\frac{d^2}{dx^2}$

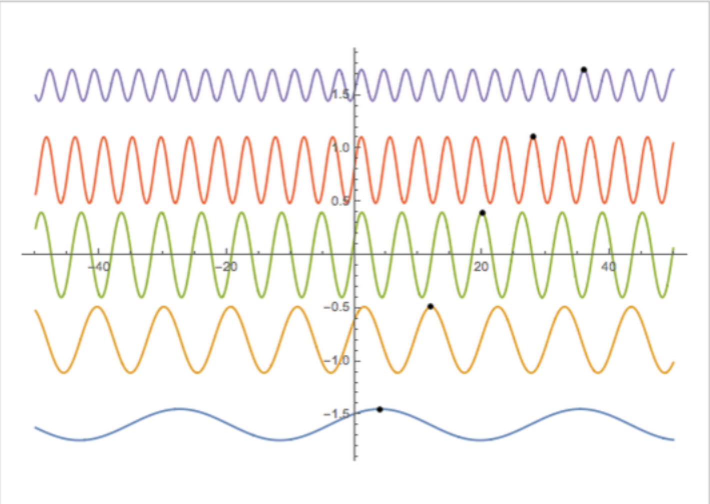
We replace M with $\frac{d^2}{dx^2}$ and v with a function $\varphi_\lambda(x)$

$$\frac{d^2}{dx^2}\varphi_\lambda = \lambda\varphi_\lambda$$

- φ_λ is called an *eigenfunction* with *eigenvalue* λ

Eigenvalues, eigenvectors, and eigenfunctions

$$\left(-\frac{d^2}{d^2x}\right)\varphi_\lambda = \lambda\varphi_\lambda, \quad \varphi_\lambda = e^{i\sqrt{\lambda}x}, \quad \sqrt{\lambda} = \text{ frequency}$$



Eigenfunctions on surfaces

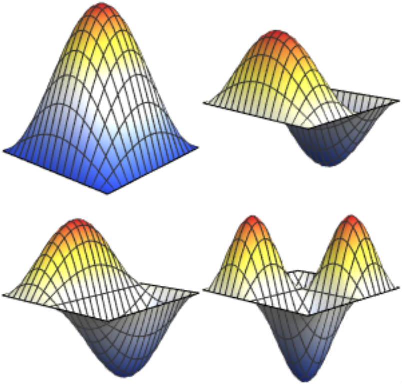
Goal: Modeling physical phenomena on surfaces more complicated than a line

Eigenfunctions on surfaces

- Given a surface S, the *Laplacian* denoted by Δ_S plays the role of $\frac{d^2}{dx^2}$.

The Laplacian

$(\Delta_S f)(x)$ describes the degree to which x is a minima of f on the surface S



Eigenfunctions of Δ_S

Given a surface S, we study eigenfunctions of Δ_S

$$\Delta_S \varphi_\lambda = \lambda \varphi_\lambda$$

- φ_λ is called a *Laplace eigenfunction* with *eigenvalue* λ

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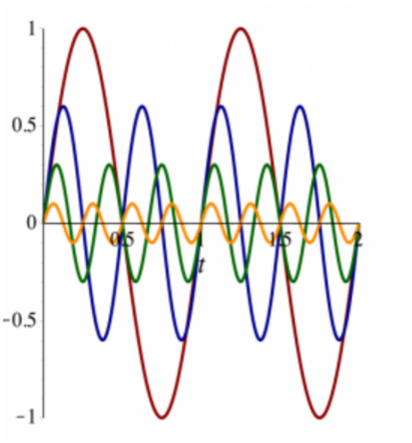
- N.B. On a surface S, λ are discrete!

Eigenfunctions on surfaces

Laplace eigenfunctions on the circle S^1 are sinusoids

$$\Delta_{S^1}\varphi_{n^2} = n^2\varphi_{n^2}, \quad \varphi_{n^2}(\theta) = e^{in\theta}$$

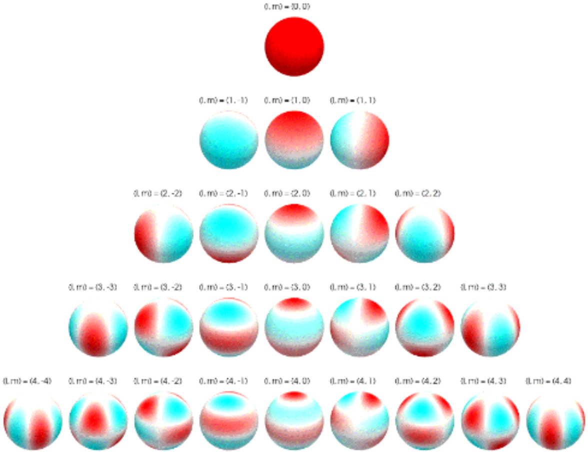


Eigenfunctions on surfaces

Visualizations of Laplace eigenfunctions on the sphere, S^2

$$\Delta_{S^2}\varphi_\lambda = \lambda\varphi_\lambda$$

Red = Positive, Blue = Negative, White = Negligible(Zero)

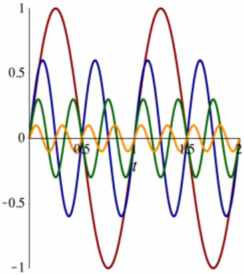


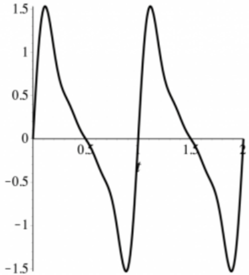
How Laplace Eigenfunctions help model quantum particles on surfaces

Theorem

Any function f, on the surface S is a superposition of Laplace eigenfunctions.

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_{\lambda_i}(x)$$





How Laplace Eigenfunctions help model quantum particles on surfaces

Quantum Particle on S

$|u(x,t)|^2$ probability distribution of locating a particle at position x and time t. Its **wavefunction** $u(x,t)$ evolves by

$$i\frac{\partial}{\partial t}u(x,t) = -\Delta_S u(x,t)$$

How Laplace Eigenfunctions help model quantum particles on surfaces

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Wavefunction $u(x,t)$ written as sum of Laplace eigenfunctions φ_λ

$$u(x,t) = \sum_{i=1}^{\infty} c_i e^{i\lambda_i t}\varphi_{\lambda_i}$$

Probability Distributions of position and direction

- u a wavefunction. $|u(x)|^2$ a probability distribution of **location**.

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- When S is a line we take u's Fourier Transform

$$\hat{u}(\xi) = \int_{\mathbf{R}} e^{-ix\cdot\xi} u(x) dx$$

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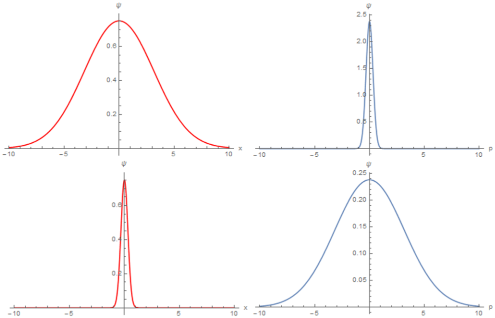
- $|\hat{u}(\xi)|^2$ a probability distribution of **direction**.



Probability Distributions of position and direction

- Accurate measurements of position and direction distributions necessary for identifying quantum states
- Setting up different experiments for measuring position and direction can be costly/impractical

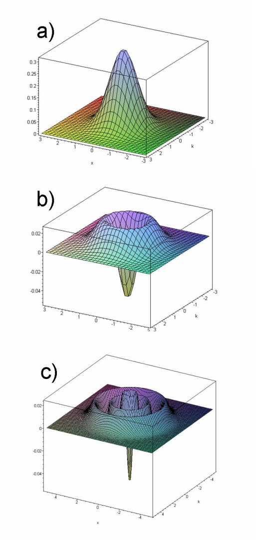
Joint Distributions of position and momentum

- $|u(x)|^2$ position distribution, $|\hat{u}(\xi)|^2$direction distribution
- Eugene Wigner: An **experimentally accessible** joint distribution $P_u(x, \xi)$

$$\int P_u(x, \xi) dx = |\hat{u}(\xi)|^2, \quad \int P_u(x, \xi) d\xi = |u(x)|^2$$

Joint Distributions of position and momentum

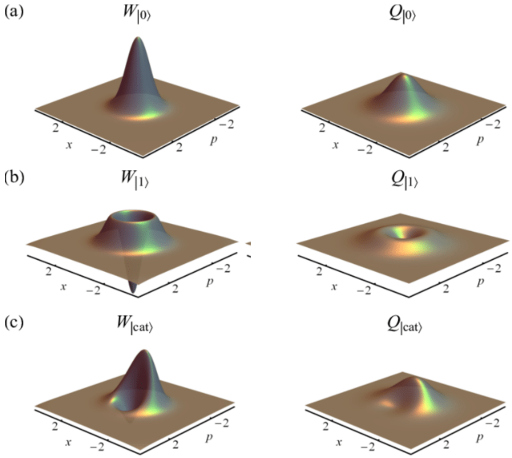
$P_u(x, \xi)$ usually not a true joint distibution because of negative regions



Joint Distributions of position and momentum

Kodi Husimi: Introduce nonnegative distribution H_u by smoothing P_u

$$H_u = P_u * G(z), \quad G(z) = e^{-|z|^2}$$



Joint Distributions of position and momentum

- For wavefunction u, Husimi Distribution H_u

$$\int H_u(x,\xi)dx \neq |\hat{u}(\xi)|^2, \quad \int H_u(x,\xi)d\xi \neq |u(x)|^2$$

- Still experimentally accessible, "nice" mathematical properties for studying regions where $|u|^2 = 0$

Joint Distributions of position and momentum

Mathematical definitions of these joint distributions rely on symmetries of flat space NOT present on a general surface

My work - an overview

For a surface S and Laplace eigenfunctions

$$\Delta_S \varphi_\lambda = \lambda \varphi_\lambda$$

Project 1

- Introducing a direct analogue of smoothed joint position and direction distributions of φ_λ and an approximate formula for their averages

Project 2

- Quantifying their rate of growth of $\varphi_\lambda(x)$ as $\lambda \to \infty$ and what this tells us about S.

Constructing a complex phase space on a surface

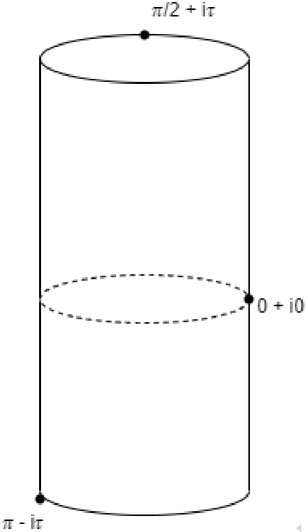
Recipe for joint position direction distributions of eigenfunctions φ_λ on a surface S

- Construct a position and direction space for S using complex numbers.
- Complex numbers allows "continuation" of Laplace eigenfunctions from S to constructed position/direction space.

Will be done for a circle, but the construction for any shape is similar.

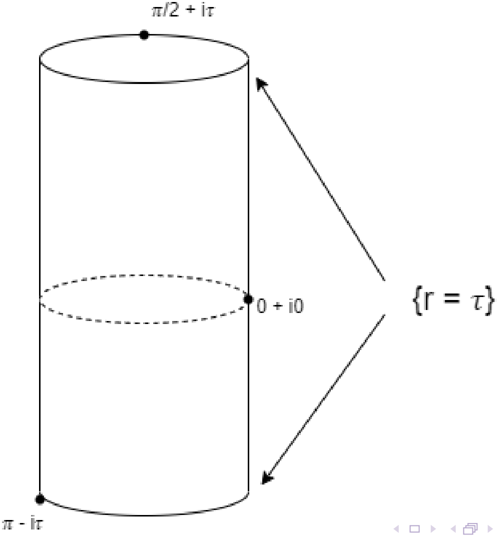
Constructing a complex phase space on a surface

We start with a circle parameterized counterclockwise, and simply add an imaginary component $i\tau$



Constructing a complex phase space on a surface

$r(\theta + i\tau) = |\tau|$ is a radius function describing how far into the cylinder we go.



Phase space distribution of eigenfunctions

"Continue" Laplace eigenfunctions φ_λ to position/direction functions.

Husimi Distributions on a Circle

For a circle

$$e^{ik\theta} \to e^{ik(\theta + i\tau)} = e^{-k\tau} e^{ik\theta}$$

Phase space distribution of eigenfunctions

"Continue" Laplace eigenfunctions φ_λ to position/direction functions.

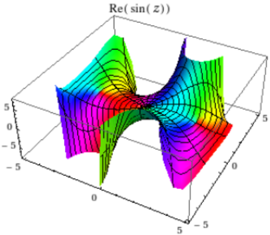
Husimi Distributions on a general surface

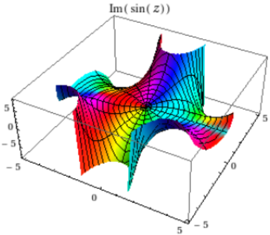
On ANY surface we analytically continue φ_λ in the same way

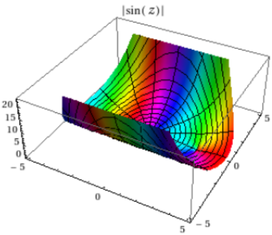
$$\varphi_\lambda(x) \to \varphi_{H,\lambda}(x+iy)$$

Phase space distribution of eigenfunctions

Cross section of position/direction distribution for a sphere eigenfunction







A result on Husimi Distributions

For $\varphi_{\lambda,H}$ a position/direction distribution of an eigenfunction

- Arbitrary $\varphi_{\lambda,H}$ intractable
- We average them instead

$$\frac{1}{\#\{\lambda_j < \lambda\}} \sum_{\lambda_j<\lambda} \varphi_{\lambda_j,H}(u)\overline{\varphi}_{\lambda_j,H}(v)$$

These are overlaps, note when $u = v$ we get $|\varphi_{\lambda_j,H}(u)|^2$

Chang–R. '21: Scaling asymptotics

For u,v within $\sqrt{\lambda}$ and position/direction eigenfunction distributions $\varphi_{\lambda,H}$

$$\frac{1}{\#\{\lambda_j < \lambda\}} \sum_{\lambda_j < \lambda} \varphi_{\lambda_j,H}(u)\overline{\varphi}_{\lambda_j,H}(v) = C \cdot e^{\left(-\frac{|u|^2}{2} - \frac{|v|^2}{2} + v\cdot\overline{u}\right)}$$

$$+ \text{ increasingly negligible terms}$$

Gaussian term permeates similar calculations in flat space

End of Project 1

Before we move to an application of project 1 are there questions?

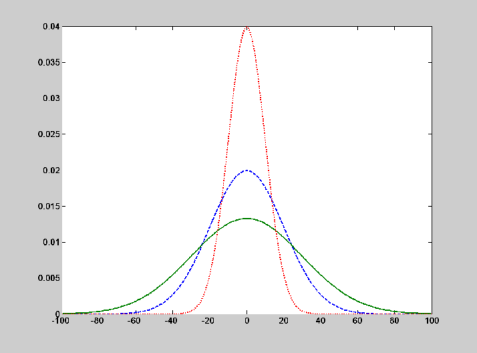
Project 2 - Concentration of Husimi Distributions

- When do eigenfunctions peak, and why?
- How does this peaking relate to the geometry of the surface?

Eigenfunction peaking

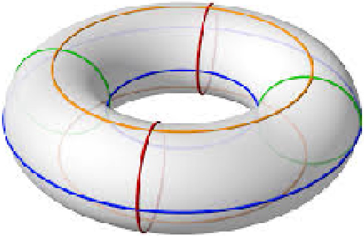
- Given a point x on a surface, how fast does eigenfunction $\varphi_\lambda(x)$ grow as a function of eigenvalue λ

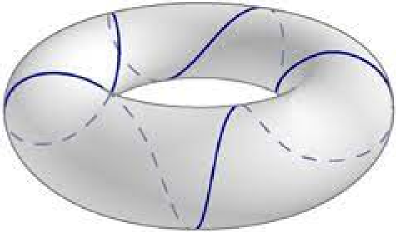
$$|\varphi_\lambda(x)| \leq f(\lambda), \quad \text{What is f?}$$



Straight Line Trajectories (geodesics)

- Point,direction on surface traces out a straight line trajectory (called a geodesic)





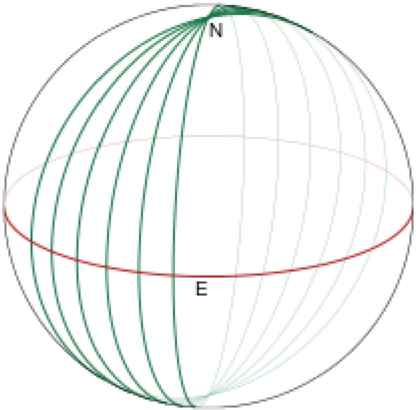
Straight Line Trajectories (geodesics)

Two important types of geodesics both found on sphere

- Periodic Geodesics
- Stable Geodesics

Straight Line Trajectories (geodesics)

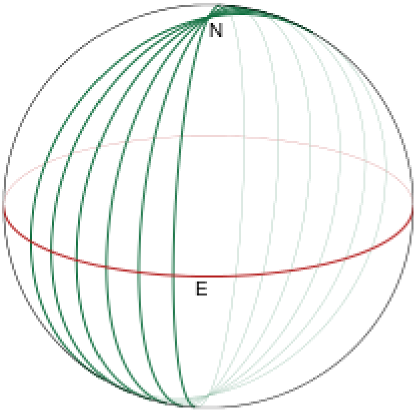
Periodic geodesics loop back to where they started from



Every geodesic starting at the north pole is periodic

Straight Line Trajectories (geodesics)

Stable geodesics don't change much when perturbed



Slight perturbations of a great circle remains close

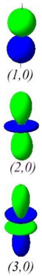
Peaking of eigenfunctions

Lars Hormander- Eigenfunction Upper Bounds

For eigenfunction φ_λ on a surface of dimension n

$$\sup_x |\varphi_\lambda(x)| \leq C\lambda^{\frac{n-1}{2}}$$

Equality occurs for spherical eigenfunctions peaking at points with many periodic geodesics!



Peaking of eigenfunctions

Chris Sogge - Eigenfunction Moment Upper Bounds

For eigenfunctions φ_λ, constant c_n depending on dimension n of surface

$$\left(\int_S |\varphi_\lambda(x)|^q dx\right)^{\frac{1}{q}} \leq \begin{cases} C\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})}, & 2 \leq q \leq c_n \\ C\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}, & c_n \leq q \end{cases}$$

- $q = \infty$: Largest value at a point.
- q small: Average value over more diffuse region

Peaking of eigenfunctions

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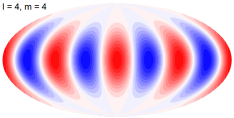
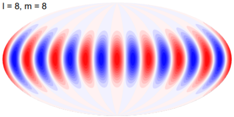
Small moment regime saturated by eigenfunctions localized to stable equator





Peaking of eigenfunctions

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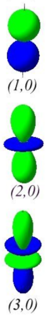
Large moment regime saturated by eigenfunctions localized to points with many periodic geodesics



My Result - Peaking of norms of Husimi Distributions

Chang-R '22: Peaking of Phase Space Norms

For $\varphi_{H,\lambda}$ position/direction distributions of eigenfunctions on a surface of dimension n

$$\left(\int |\varphi_{H,\lambda}(x)|^q dx\right)^{\frac{1}{q}} \leq C\lambda^{(n-1)(\frac{1}{2}-\frac{1}{q})} \qquad (2 \leq q \leq \infty).$$

My Result - Peaking of norms of Husimi Distributions

Chang-R '22: Peaking of Phase Space Norms

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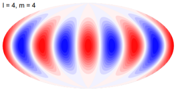
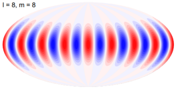
Saturated by position/direction distributions of of eigenfunctions localized to stable geodesics!





Thank you for coming!